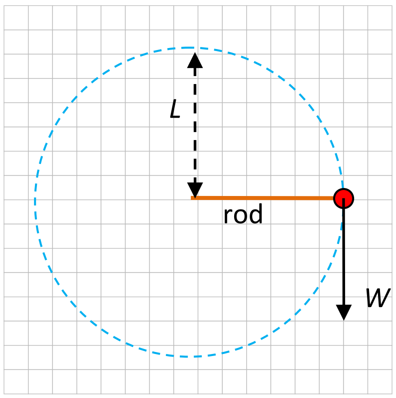
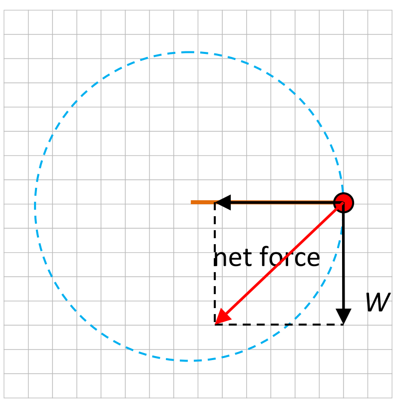
Teacher notes
Topic A

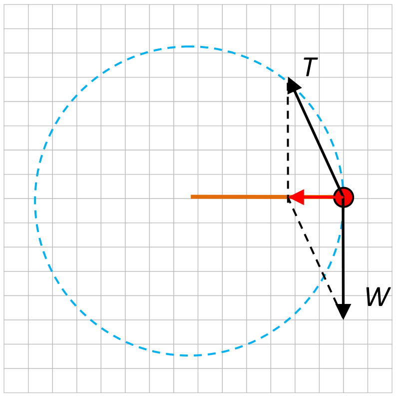
A rod of length *L* is attached to a ball that rotates on a vertical circle with constant speed. What is the direction of the force on the ball due to the rod at the position shown?



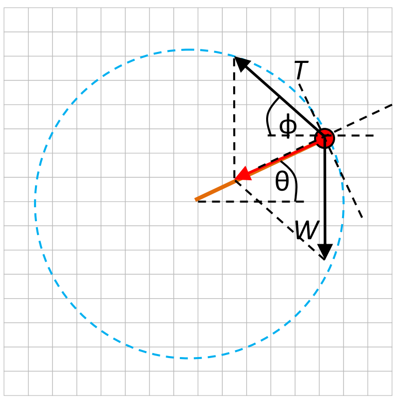
This is a confusing question; most students would go for a horizontal force along the rod pointing to the center of the circle. But this is not correct. If this were the case the net force would **not** be pointing towards the center!



The answer must be a force *T* as shown. The vertical component of this force is equal to the weight leaving the horizontal component as the net force which **does** point towards the center as it should.



At some arbitrary position we have the rod making an angle θ to the horizontal:



Getting components of forces along the direction of the rod and normally to it we find:

$T\sin\phi = W\cos\theta$ and $T\cos\phi + W\sin\theta = \frac{mv^2}{L}$.

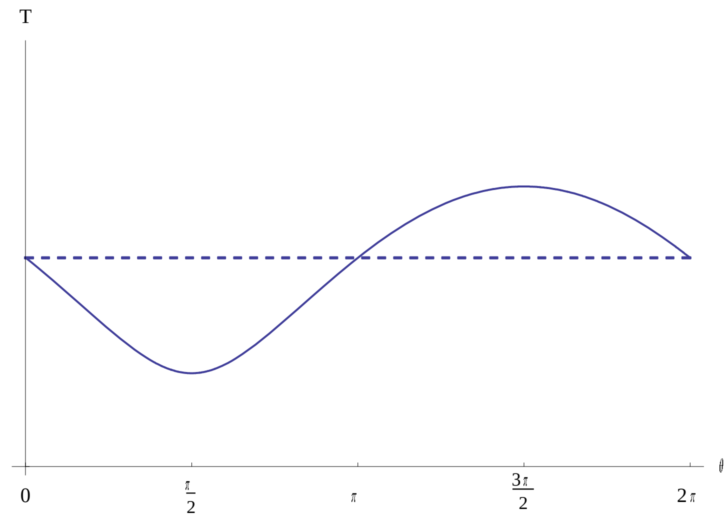
This means

$$T^2\sin^2\phi = W^2\cos^2\theta$$
$$T^2\cos^2\phi = \left(\frac{mv^2}{L} - W\sin\theta\right)^2$$

and so adding we get

$$T^2 = W^2 + (\frac{mv^2}{L})^2 - 2\frac{mv^2}{L}W\sin\theta$$

This shows that the tension force *T* in the rod is **not** constant since it depends on θ:



The maximum occurs when $\sin\theta = -1$, i.e. at the lowest point in the circle and the minimum at $\sin\theta = +1$, i.e. at the top of the circle. Clearly,

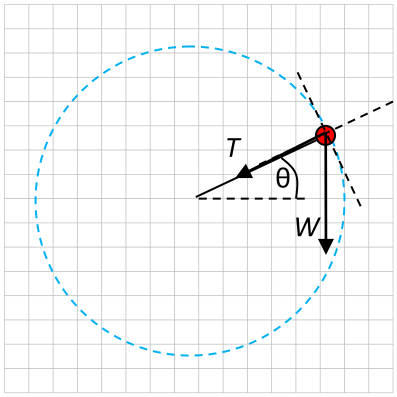
$$T_{\max}^2 = W^2 + (\frac{mv^2}{L})^2 + 2\frac{mv^2}{L}W = (\frac{mv^2}{L} + W)^2 \Rightarrow T_{\max} = \frac{mv^2}{L} + W$$

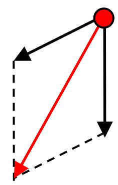
$$T_{\min}^2 = W^2 + (\frac{mv^2}{L})^2 - 2\frac{mv^2}{L}W = (\frac{mv^2}{L} - W)^2 \Rightarrow T_{\min} = \frac{mv^2}{L} - W$$

At $\theta = 0$ (i.e. the rod is horizontal) : $T = \sqrt{W^2 + (\frac{mv^2}{L})^2}$.

Students will often ask why the force is not along the rod but a diagram like the one below showing the connection of the rod to the ball can answer this question.

Finally, what happens if the rod is replaced by a string? In this case the tension force **is** along the string!





This means the net force of the tension and the weight is somewhere in between *T* and *W*. How can this be? This is because the **speed is not constant**. The ball is accelerating not only because of the centripetal acceleration but also due to a linear acceleration that changes the speed:

$T + W\sin\theta = \frac{mv^2}{L}$ and $W\cos\theta = ma\ (= m\frac{dv}{dt})$.

So the ball has two accelerations: the centripetal acceleration towards the center and one linear acceleration that is tangential to the circle. Adding these accelerations would give a net force in the direction of the red arrow, i.e. the net force.

The only places where the tangential acceleration is zero is at the top ($\theta = \frac{\pi}{2}$) and bottom ($\theta = -\frac{\pi}{2}$) of the path and in that case we get the familiar results that:

$$T_{\text{top}} = \frac{mv_{\text{top}}^2}{L} - W$$

$$T_{\text{bottom}} = \frac{mv_{\text{bottom}}^2}{L} + W$$

Conservation of energy says that $\frac{1}{2}mv_{\text{bottom}}^2 = \frac{1}{2}mv_{\text{top}}^2 + 2mgL$ so that we now get the other familiar result that:

$$\begin{aligned} T_{\text{bottom}} - T_{\text{top}} &= \left(\frac{mv_{\text{bottom}}^2}{L} + W\right) - \left(\frac{mv_{\text{bottom}}^2 - 4mgL}{L} - W\right) \\ &= 4mg + 2W \\ &= 6mg \end{aligned}$$